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On the rank of the intersection of free subgroups in virtually free groups

We prove an estimate for the rank of the intersection of free subgroups in virtually free groups, which is analogous to the Hanna Neumann inequality for subgroups in a free group and to the S.V. Ivanov estimate for subgroups in free products of groups. We also prove a more general estimate for the rank of the intersection of free subgroups in the fundamental group of a finite graph of groups with finite edge groups.

1. Introduction.

Suppose first G is a free group, H and K are finitely generated subgroups in G. In 1954 Howson [1] proved that in this case subgroup $H \cap K$ is also finitely generated. Then in 1957 Hanna Neumann [2] proved the following estimate for the rank of intersection of subgroups in a free group (*Hanna Neumann inequality*):

$$\overline{r}(H \cap K) \leqslant 2\,\overline{r}(H)\overline{r}(K), \tag{1}$$

where $\overline{r}(H) = max\,(r(H)-1, 0)$ is the reduced rank of subgroup H, $r(H)$ is the rank of subgroup H.

In 2011 Igor Mineyev [3] proved the famous Hanna Neumann conjecture which states that the coefficient 2 in the inequality (1) can be omitted:

$$\overline{r}(H \cap K) \leqslant \overline{r}(H)\overline{r}(K).$$

S.V.Ivanov proved an estimate for subgroups of free products of groups, which is analogous to the Hanna Neumann inequality. Namely, in 1999 S.V.Ivanov [4] proved that, if $G = G_1 * G_2$ is a free product of groups, H and K are finitely generated subgroups in G which intersect trivially with all the conjugates to the factors G_1 and G_2 (therefore, according to Kurosh subgroup theorem [5], H and K are free), then the intersection $H \cap K$ is also finitely generated and the following estimate holds:

$$\overline{r}(H \cap K) \leqslant 6\,\overline{r}(H)\overline{r}(K), \tag{2}$$

Later S.V.Ivanov and W.Dicks [6] proved a more precise estimate for subgroups of free products which generalizes the estimate (2), and S.V.Ivanov [7] proved an analogous bound for the Kurosh rank of (arbitrary) subgroups of a free product.

The author [8] proved an estimate for the rank of the intersection of free subgroups in free products of groups amalgamated over a finite normal subgroup, this estimate generalizes the inequality (2) and the estimate proved by S.V.Ivanov and W.Dicks in [6].

In this article we prove an estimate which generalizes the inequality (2) to the case of subgroups of the fundamental group of a finite graph of groups with finite edge groups. Estimates for the rank of the intersection of subgroups in free products of groups amalgamated over a finite subgroup, as well as subgroups in HNN-extensions of groups with finite associated subgroups, follow as corollaries. Another corollary, which is obtained by applying a theorem of Stallings, is an estimate for the rank of the intersection of free subgroups in virtually free groups.

2. Bass-Serre theory.

In this article we use Bass-Serre theory of groups acting on trees. The main facts from this theory which we use are described below. More detailed description of this theory can be found in [9], [10].

First we remind some definitions from the graph theory and fix the notations.

Graphs, quotient graphs

A graph X is a tuple consisting of a nonempty set of vertices $V(X)$, a set of edges $E(X)$ and three mappings: $\alpha : E(X) \to V(X)$ (beginning of an edge), $\omega : E(X) \to V(X)$ (end of an edge) and ${}^{-1} : E(X) \to E(X)$ (inverse edge) such that $(e^{-1})^{-1} = e$, $e^{-1} \neq e$, $\alpha(e) = \omega(e^{-1})$ for every $e \in E(X)$. A graph is called finite if the sets of its edges and vertices are finite. The notion of a subgraph can be defined in a natural way. A morphism from a graph X to a graph Y is a map p from the set of vertices and edges of X to the set of vertices and edges of Y which sends vertices to vertices, edges to edges and such that $\alpha(p(e)) = p(\alpha(e))$, $\omega(p(e)) = p(\omega(e))$, $p(e^{-1}) = (p(e))^{-1}$. A bijective morphism of graphs is called an isomorphism. The degree of a vertex $v \in V(X)$ is the number of edges of the graph X beginning in v (notation: $deg\, v$). A morphism of graphs is called locally injective if it sends every two different edges beginning in the same vertex to different edges.

A graph is called oriented if in each pair of its mutually inverse edges e, e^{-1} one edge is chosen and called positively oriented; the other edge is called negatively oriented. The set of all positively oriented edges of the graph X will be denoted as $E(X)^+$, and the set of all negatively oriented edges — as $E(X)^-$. If graphs X and Y are oriented and p is a morphism from the graph X to the graph Y, which sends positively oriented edges of X to positively oriented edges of Y, we will say that p preserves orientation. While constructing some graphs below we will define only their positively oriented edges, the negatively oriented edges are then defined in a natural way as the inverse edges to the (corresponding) positively oriented edges.

A sequence $l = e_1e_2...e_n$ of edges of a graph X is called a path beginning in $\alpha(e_1)$ and ending in $\omega(e_n)$ if $\omega(e_i) = \alpha(e_{i+1})$, $i = 1, ..., n-1$. (We assume that any vertex v of X is also a path beginning and ending in v, which we call trivial path at v.) A path is called reduced if it does not contain subpaths of the form dd^{-1}, where $d \in E(X)$. A path is called cyclically reduced if it is reduced and its first edge does not coincide with the inverse to its last edge; a trivial path is also cyclically reduced. A path is closed if its beginning and end coincide. A graph X is called connected if for any two of its vertices u and v there exists a path in X beginning in u and ending in v. A tree is a connected graph which has no nontrivial reduced closed paths. A maximal subtree of a connected graph X is a subtree which is maximal with respect to inclusion; it is easy to see that a maximal subtree of X contains all vertices of X. The image of a path under a morphism of graphs is defined in a natural way: $p(e_1e_2...e_n) = p(e_1)p(e_2)...p(e_n)$.

Suppose X is a connected graph with a distinguished vertex x. Two closed paths p_1 and p_2 in X beginning in x are called homotopic if p_2 can be obtained from p_1 by a finite number of insertions and deletions of subpaths of the form ee^{-1}, $e \in E(X)$. One can see that the set of all reduced closed paths in X beginning in x forms a group with respect to the following multiplication: the product of two reduced paths $e_1e_2...e_n$ and $f_1f_2...f_m$ ($e_i, f_j \in E(X)$, $i = 1, ..., n$, $j = 1, ..., m$) is the unique reduced closed path beginning in x which is homotopic to the path $e_1e_2...e_nf_1f_2...f_m$; the identity of this group is the trivial path at the vertex x; the inverse to the path $l = e_1...e_n$ is the path $l^{-1} = e_n^{-1}...e_1^{-1}$. This group is called the fundamental group of the graph X with respect to the vertex x and is denoted by $\pi_1(X, x)$. It is easy to see that the fundamental group of a connected graph X does not depend on the choice of a distinguished vertex in X up to isomorphism. The isomorphism class of the fundamental group of X is denoted by $\pi_1(X)$.

The fundamental group of every connected graph X is free. Moreover, let S be a maximal

subtree in X, r_v (for each $v \in V(X)$) denote the unique reduced path beginning in x and ending v which lies in S, $q_e = r_{\alpha(e)} e r_{\omega(e)}^{-1}$ (for each $e \in E(X)$). Suppose X is oriented (and thus S is oriented as well). Then one can prove that the paths q_e, $e \in E(X)^+ - E(S)^+$ are free generators of the group $\pi_1(X, x)$ (see [9]).

Suppose the graph X is finite. Then the following holds:

$$r(\pi_1(X, x)) = |E(X)^+| - |E(S)^+| = |E(X)^+| - |V(X)| + 1, \tag{3}$$

where the last equality holds since S is a tree which contains all vertices of X.

We say that a group G acts on a graph X on the left if left actions of G on the sets $V(X)$ and $E(X)$ are defined so that $g\alpha(e) = \alpha(ge)$, $g\omega(e) = \omega(ge)$ and $ge^{-1} = (ge)^{-1}$ for all $g \in G$, $e \in E(X)$. We say that G acts on X without inversion of edges if $ge \neq e^{-1}$ for all $e \in E(X)$, $g \in G$.

Let G be a group acting on a graph X without inversion of edges. For $x \in V(X) \cup E(X)$ denote by $Orb\,(x)$ the orbit of x with respect to the action of G: $Orb\,(x) = \{gx,\ g \in G\}$. Define the *quotient graph* $G \setminus X$ (or $X\ /\ G$) as the graph with vertices $Orb\,(v)$, $v \in V(X)$ and edges $Orb\,(e)$, $e \in E(X)$; $Orb\,(v)$ is the beginning of $Orb\,(e)$ (in $G \setminus X$) if there exists $g \in G$ such that gv is the beginning of e (in X); the inverse of the edge $Orb\,(e)$ is the edge $Orb\,(e^{-1})$.

Notice that the edges $Orb\,(e)$ and $Orb\,(e^{-1})$ do not coincide since G acts on X without inversion of edges. It is easy to see that the map $p:\ X \to G \setminus X,\ \ p(x) = Orb\,(x)$, $x \in V(X) \cup E(X)$, is a surjective morphism of graphs; we call it the projection on the quotient graph.

The fundamental group of a graph of groups

A *graph of groups* (Γ, Y) consists of a connected graph Y, vertex groups G_v for each vertex $v \in V(Y)$, edge groups G_e for each edge $e \in E(Y)$ such that $G_e = G_{e^{-1}}$ for all $e \in E(Y)$, and group embeddings $\alpha_e : G_e \to G_{\alpha(e)}$, $e \in E(Y)$. One can also define the group embeddings $\omega_e : G_e \to G_{\omega(e)}$, $\omega_e = \alpha_{e^{-1}}$. The graph of groups (Γ, Y) is called finite if the graph Y is finite. The graph of groups (Γ, Y) is called a graph of finite groups if all the vertex groups (and, therefore, all the edge groups as well) of (Γ, Y) are finite.

Let S be a maximal subtree of the graph Y. The *fundamental group of the graph of groups* (Γ, Y) with respect to the maximal subtree S (notation $\pi_1(\Gamma, Y, S)$) is the quotient group of the free product of all vertex groups G_v, $v \in V(Y)$, and the free group with basis $\{t_e,\ e \in E(Y)\}$ by the normal closure of the set of the following elements:

$$t_e^{-1}\alpha_e(g)t_e \cdot (\alpha_{e^{-1}}(g))^{-1} \quad (e \in E(Y),\ g \in G), \qquad t_e t_{e^{-1}}\ (e \in E(Y)), \qquad t_e\ (e \in E(S)).$$

One can prove (see [9]) that the fundamental group $\pi_1(\Gamma, Y, S)$ of the graph of groups (Γ, Y) does not depend on the choice of the maximal subtree S in Y up to isomorphism. Therefore we will sometimes speak about the fundamental group of a graph of groups without mentioning the maximal subtree.

Consider the following examples. Suppose Y consists of one pair of mutually inverse edges e, e^{-1} and two vertices u, v (of degree 1), then it is easy to see that the fundamental group of the graph of groups (Γ, Y) is isomorphic to the free product of groups G_u and G_v amalgamated over the subgroup $\alpha_e(G_e) = \omega_e(G_e)$.

Suppose Y consists of one pair of mutually inverse edges e, e^{-1} and one vertex u (of degree 2), then it is easy to see that the fundamental group of the graph of groups (Γ, Y) is isomorphic to the HNN-extension with base group G_u and associated subgroups $\alpha_e(G_e)$ and $\omega_e(G_e)$.

We fix an arbitrary orientation of Y, and thus suppose below that Y (and, therefore, S as well) is oriented.

Notice that if (Γ, Y) is an arbitrary finite graph of groups and S is a maximal subtree in Y, then the group $\pi_1(\Gamma, Y, S)$ can be obtained by successive applications of amalgamated

free product construction (corresponding to the positively oriented edges of S), followed by successive applications of HNN-extension construction (corresponding to the positively oriented edges of Y, not belonging to S).

One can see that if all the vertex groups (and, therefore, all the edge groups as well) of (Γ, Y) are trivial then the fundamental group of the graph of groups (Γ, Y) is isomorphic to the fundamental group of Y, in particular, this group is free. Indeed, it follows from the definition of the fundamental group of a graph of groups that $\pi_1(\Gamma, Y, S)$ is free in this case; furthermore, its rank is equal to the number of positively oriented edges of Y not belonging to the maximal subtree S of Y, and the rank of $\pi_1(Y)$ is equal to the same number, as mentioned above.

One can prove (see [9]) that the vertex groups G_v, $v \in V(Y)$, can be canonically embedded in the group $\pi_1(\Gamma, Y, S)$. We identify the group G_v with its canonical image in the group $\pi_1(\Gamma, Y, S)$ for each $v \in V(Y)$. Moreover, we identify the group G_e with the canonical image of the subgroup $\alpha_e(G_e)$ in the group $\pi_1(\Gamma, Y, S)$ for each $e \in E(Y)^+$.

Bass-Serre theorem

The following theorem shows the connection between fundamental groups of graphs of groups and groups acting on trees (without inversion of edges). More details and the proof of this theorem can be found in [9] and [10].

Theorem (Bass, Serre). (1) *Let $G = \pi_1(\Gamma, Y, S)$ be the fundamental group of a graph of groups (Γ, Y) (with respect to a maximal subtree S). Then the group G acts without inversion of edges on some tree T so that*

1. *The quotient graph $G \setminus T$ is isomorphic to the graph Y.*
2. *The stabilizer of a vertex v under the action of G is conjugate to the vertex group $G_{p(v)}$ of the graph of groups (Γ, Y) for every $v \in V(T)$.*
3. *The stabilizer of an edge e under the action of G is conjugate to the edge group $G_{p(e)}$ of the graph of groups (Γ, Y) for every $e \in E(T)$.*

(Here $p : T \to G \setminus T$ is the projection on the quotient graph; due to the condition 1 we can identify the graphs Y and $G \setminus T$ and assume that $p : T \to Y$.)

(2) *Conversely, let G be a group acting without inversion of edges on a tree T. Then the group G is isomorphic to the fundamental group $\pi_1(\Gamma, Y, S)$ of some graph of groups (Γ, Y) such that the conditions 1, 2, 3 from the first part of the theorem hold. In particular (since p is surjective), each vertex group of (Γ, Y) is equal to the stabilizer of some vertex of T and each edge group of (Γ, Y) is equal to the stabilizer of some edge of T.*

Bass-Serre tree

Suppose $G = \pi_1(\Gamma, Y, S)$. We need the explicit construction of the tree T and the action of G on T from the first part of Bass-Serre theorem. This tree T constructed below is called *Bass-Serre tree.*

If $H \subseteq G$ then we denote by G/H the set of all left cosets of H in G.

The graph T can be defined as follows (it is oriented; the unions below are disjoint):

- Vertices of T are the left cosets of the vertex groups of (Γ, Y) in G:

$$V(T) = \bigcup_{v \in V(Y)} G/G_v.$$

- Positively oriented edges of T are the left cosets of the edge groups of (Γ, Y) in G:

$$E(T)^+ = \bigcup_{e \in E(Y)^+} G/G_e.$$

- The following equalities hold:

$$\alpha(gG_e) = gG_{\alpha(e)}, \quad \omega(gG_e) = gt_eG_{\omega(e)}, \quad g \in G,\ e \in E(Y)^+. \tag{4}$$

(Remind that we identify the edge group G_e with the canonical image of subgroup $\alpha_e(G_e)$ in the group G for every $e \in E(Y)^+$; $t_e = 1$ whenever e lies in the maximal subtree S)

One can prove (see [9]) that T is a tree.

Define the left action of G on T as action by the left multiplication:

$$g_1 \cdot (gG_v) = g_1gG_v, \quad g_1 \cdot (gG_e) = g_1gG_e, \quad g, g_1 \in G,\ v \in V(Y),\ e \in E(Y)^+.$$

(The action of G on negatively oriented edges of T is defined in a natural way: $g \cdot f = (g \cdot f^{-1})^{-1}$, $f \in E(T)^-$, $g \in G$.) It is easy to see that it is an action of G on T without inversion of edges.

Notice that the conditions 1, 2, 3 from Bass-Serre theorem hold for T.

Subgroups of the fundamental group of a graph of groups which intersect trivially with the conjugates to the vertex groups

Suppose $G = \pi_1(\Gamma, Y, S)$ and $H \subseteq G$ is a subgroup which intersects trivially with the conjugates to all the vertex groups (and, therefore, all the edge groups as well) of (Γ, Y). According to the first part of Bass-Serre theorem, the group G acts on a tree T (without inversion of edges) so that the conditions 1, 2, 3 from the theorem hold, and the structure of T is described above (Bass-Serre tree).

Therefore, the group H also acts on the tree T in a natural way (we restrict the action of G to its subgroup H). Let $v \in V(T)$. Denote by $Stab_G\,(v)$ the stabilizer of a vertex v under the action of G on T, and by $Stab_H\,(v)$ — the stabilizer of a vertex v under the action of H on T. Notice that $Stab_H\,(v) = Stab_G\,(v) \cap H = \{1\}$. The last equality holds since, according to the condition 2 from Bass-Serre theorem, subgroup $Stab_G\,(v)$ is conjugate to some vertex group of (Γ, Y), and subgroup H intersects trivially with all the conjugates to the vertex groups.

Thus the group H acts on the tree T (without edge inversions) so that the stabilizers of all vertices (and, therefore, of all edges as well) of T are trivial. According to the second part of Bass-Serre theory, we obtain that $H \cong \pi_1(\Gamma', Y', S')$, where (Γ', Y') is a graph of groups, S' is a maximal subtree in Y' and Y' is isomorphic to $H \setminus T$ (due to the condition 1), and all vertex groups of (Γ', Y') are equal to the stabilizers of some vertices of T under the action of H, and thus are trivial. Therefore, as mentioned above

$$H \cong \pi_1(\Gamma', Y', S') \cong \pi_1(Y') \cong \pi_1(H \setminus T),$$

in particular, H is free.

Thus if subgroup $H \subseteq \pi_1(\Gamma, Y, S)$ intersects trivially with the conjugates to all the vertex groups of (Γ, Y) then H is free and, moreover,

$$H \cong \pi_1(H \setminus T), \tag{5}$$

where T is the Bass-Serre tree for G.

If $H_1, H_2 \subseteq G$ then we denote by $H_1 \backslash G / H_2$ the set of all double cosets of H_1 and H_2 in G (of the form H_1gH_2, $g \in G$).

Notice that, according to the structure of the Bass-Serre tree T, the graph $H \setminus T$ has the following structure (it is oriented; the unions below are disjoint):

- Vertices of $H\backslash T$ are the double cosets of the following form:

$$V(T) = \bigcup_{v \in V(Y)} H\backslash G/G_v.$$

- Positively oriented edges of $H\backslash T$ are the double cosets of the following form:

$$E(T)^+ = \bigcup_{e \in E(Y)^+} H\backslash G/G_e.$$

- The following equalities hold (which follow from the equalities (4)):

$$\alpha(HgG_e) = HgG_{\alpha(e)}, \quad \omega(HgG_e) = Hgt_eG_{\omega(e)}, \quad g \in G,\ e \in E(Y)^+. \tag{6}$$

3. The main results.

Theorem 1. *Suppose G is the fundamental group of a finite graph of groups (Γ, Y) with finite edge groups, $H, K \subseteq G$ are finitely generated subgroups which intersect trivially with the conjugates to all the vertex groups of (Γ, Y) (and are, therefore, free). Then the following estimate holds:*

$$\overline{r}(H \cap K) \leqslant 6m \cdot \overline{r}(H)\,\overline{r}(K), \tag{7}$$

where m is the maximum of the orders of the edge groups of (Γ, Y).

(Remind that $\overline{r}(H) = max\,(r(H) - 1, 0)$ is the reduced rank of subgroup H.)

Applying Theorem 1 in the case when Y consists of one pair of mutually inverse edges and two vertices, we obtain the following corollary.

Corollary 1. *Let G be a free product of two groups amalgamated over a finite subgroup of order m, $H, K \subseteq G$ be finitely generated subgroups which intersect trivially with the conjugates to the factors of G (and are, therefore, free). Then the following estimate holds:*

$$\overline{r}(H \cap K) \leqslant 6m \cdot \overline{r}(H)\,\overline{r}(K).$$

Applying Theorem 1 in the case when Y consists of one pair of mutually inverse edges and one vertex, we obtain the following corollary.

Corollary 2. *Let G be an HNN-extension with finite associated subgroups of order m, $H, K \subseteq G$ be finitely generated subgroups which intersect trivially with the conjugates to the base group of G (and are, therefore, free). Then the following estimate holds:*

$$\overline{r}(H \cap K) \leqslant 6m \cdot \overline{r}(H)\,\overline{r}(K).$$

A group is called virtually free if it contains a free subgroup of finite index. Remind that a graph of finite groups is a graph of groups with finite edge and vertex groups.

Theorem (Stallings, [11]). *Suppose G is a finitely generated group. Then G is virtually free if and only if G is the fundamental group of a finite graph of finite groups.*

Below we show that the following theorem follows from Theorem 1 and Stallings theorem.

Theorem 2. *Suppose G is a virtually free group, subgroups $H, K \subseteq G$ are free and finitely generated. Then the following estimate holds:*

$$\overline{r}(H \cap K) \leqslant 6n \cdot \overline{r}(H)\,\overline{r}(K), \tag{8}$$

where n is the minimal index of a free subgroup in G.

4. Proof of Theorem 2.

Here we deduce Theorem 2 from Theorem 1 and Stallings theorem.

First notice that it suffices to prove Theorem 2 for finitely generated group G. Indeed, instead of group G we can consider group $G_0 \subseteq G$ which is generated by subgroups H and K; G_0 is finitely generated since H and K are finitely generated. Notice that if $R \subseteq G$ is a subgroup of a virtually free group, then R is also virtually free and the minimal index of a free subgroup in R is less than or equal to the minimal index of a free subgroup in G. (Indeed, let $F \subseteq G$ be the free subgroup of minimal index in G, then $R \cap F \subseteq R$ is a free subgroup of finite index in R and $|R : R \cap F| \leqslant |G : F|$.) Therefore G_0 is virtually free and it suffices to prove the estimate (8) for G_0.

Thus, we can suppose that G is finitely generated. Applying Stallings theorem, we obtain that G is a fundamental group of a finite graph of finite groups (Γ, Y). Moreover, subgroups H and K are finitely generated and intersect trivially with the conjugates to all the vertex groups of (Γ, Y) (since H and K are free, and the vertex groups of (Γ, Y) are finite). Therefore, all the conditions of Theorem 1 hold. Applying this theorem, we obtain that the estimate (7) holds. Thus, to prove the estimate (8) of Theorem 2 it suffices to show that the maximum of orders of edge groups of (Γ, Y) is less than or equal to the minimal index n of a free subgroup in G (which is the fundamental group of the graph of groups (Γ, Y)).

We show that, moreover, the maximum of orders of vertex groups of (Γ, Y) is less than or equal to n. Indeed, otherwise there exists a vertex group G_v of (Γ, Y) such that $|G_v| > n = |G : F|$, where the subgroup $F \subseteq G$ is free. Then there exist $g_1 \neq g_2 \in G_v : g_1F = g_2F$, therefore $1 \neq g_2^{-1}g_1 \in G_v \cap F$, and we get a contradiction since G_v is finite, and F is free. Thus, the estimate (8) holds.

5. Proof of Theorem 1.

Suppose (Γ, Y) is a finite graph of groups with vertex groups G_v ($v \in V(Y)$) and finite edge groups G_e ($e \in E(Y)$), $\alpha_e : G_e \to G_{\alpha(e)}$ and $\omega_e : G_e \to G_{\omega(e)}$ are the embeddings of the edge groups in the (corresponding) vertex groups.

Let G be the fundamental group of the graph of groups (Γ, Y), $H, K \subseteq G$ be finitely generated subgroups which intersect trivially with the conjugates to all the vertex groups of (Γ, Y).

Suppose that T is the Bass-Serre tree corresponding to the graph of groups (Γ, Y) (see Section 2).

Then, according to Bass-Serre theory, (see Section 2), subgroups H and K are free and, moreover, according to (5),

$$H \cong \pi_1(T/H), \quad K \cong \pi_1(T/K), \quad H \cap K \cong \pi_1(T/(H \cap K)). \tag{9}$$

As mentioned in Section 2, the graph T/H has the following structure: its vertices are the double cosets of the form HgG_v, $v \in V(Y)$, $g \in G$, positively oriented edges are the double cosets of the form HgG_e, $e \in E(Y)^+$, $g \in G$, and the equalities (6) hold:

$$\alpha(HgG_e) = HgG_{\alpha(e)}, \quad \omega(HgG_e) = Hgt_eG_{\omega(e)}, \quad g \in G,\ e \in E(Y)^+$$

(here we identify the edge group G_e with $\alpha_e(G_e)$ for $e \in E(Y)^+$).

We will also say that the vertex HgG_u of the graph T/H corresponds to the vertex $u \in V(Y)$, and the edge HgG_e of the graph T/H corresponds to the edge $e \in E(Y)^+$.

Notice that if an edge $Hg'G_e$ of the graph T/H begins in the vertex HgG_u ($g, g' \in G$, $e \in E(Y)^+$, $u \in V(Y)$), then, according to (6), $Hg'G_u = HgG_u$, therefore $g' = hgg_u$, $h \in H$, $g_u \in G_u$, so $Hg'G_e = Hgg_uG_e$. Thus, all positively oriented edges of the graph T/H beginning in the vertex HgG_u have the form Hgg_uG_e, where $\alpha(e) = u$, $g_u \in G_u$.

Analogously, if an edge $Hg'G_e$ of the graph T/H ends in the vertex HgG_u $(g, g' \in G,\ e \in E(Y)^+,\ u \in V(Y))$, then, according to (6), $Hg't_eG_u = HgG_u$, therefore $g' = hgg_ut_e^{-1}$, $h \in H$, $g_u \in G_u$, so $Hg'G_e = Hgg_ut_e^{-1}G_e$. Thus, all positively oriented edges of the graph T/H ending in the vertex HgG_u have the form $Hgg_ut_e^{-1}G_e$, where $\omega(e) = u$, $g_u \in G_u$.

The graphs T/K and $T/(H \cap K)$ have a similar structure.

Define the projections $\pi_H : T/(H \cap K) \to T/H$ and $\pi_K : T/(H \cap K) \to T/K$ as follows:

$$\pi_H((H \cap K)gG_u) = HgG_u,\ \pi_H((H \cap K)gG_e) = HgG_e, \quad g \in G, u \in V(Y), e \in E(Y)^+ \tag{10}$$

and similarly for K; π_H and π_K are defined on negatively oriented edges in a natural way.

It is easy to see that π_H and π_K are graph morphisms which preserve orientation.

We will now prove a few lemmas.

The following lemma shows that the projections π_H and π_K are locally injective graph morphisms.

Lemma 1. *Suppose that the conditions of Theorem 1 hold. Let $z_1 \neq z_2$ be edges of the graph $T/(H \cap K)$ such that $\alpha(z_1) = \alpha(z_2)$ or $\omega(z_1) = \omega(z_2)$. Then $\pi_H(z_1) \neq \pi_H(z_2)$ and $\pi_K(z_1) \neq \pi_K(z_2)$.*

☐ We can assume that the edges z_1 and z_2 have the same orientation (both positive or both negative) since otherwise their projections $\pi_H(z_1)$ and $\pi_H(z_2)$ also have opposite orientation (one positive and the other negative), therefore, $\pi_H(z_1) \neq \pi_H(z_2)$; similarly for π_K.

By considering if necessary z_1^{-1} and z_2^{-1} instead of z_1 and z_2 respectively, we can assume that both edges z_1 and z_2 are positively oriented.

Furthermore, we can assume that both edges z_1 and z_2 correspond to the same edge $e \in E(Y)^+$, since otherwise their projections $\pi_H(z_1)$ and $\pi_H(z_2)$ also correspond to different edges of Y, and, therefore, $\pi_H(z_1) \neq \pi_H(z_2)$; similarly for π_K.

Suppose first $\alpha(z_1) = \alpha(z_2)$. Let $\alpha(z_1) = \alpha(z_2) = v$ correspond to $u \in V(Y)$, where $\alpha(e) = u$, then, as mentioned above,

$$v = (H \cap K)gG_u, \quad z_1 = (H \cap K)gg_1G_e, \quad z_2 = (H \cap K)gg_2G_e, \quad g \in G,\ g_1, g_2 \in G_u.$$

Then, according to the definition of the projection π_H,

$$\pi_H(v) = HgG_u, \quad \pi_H(z_1) = Hgg_1G_e, \quad \pi_H(z_2) = Hgg_2G_e,$$

and $\alpha(\pi_H(z_1)) = \alpha(\pi_H(z_2)) = \pi_H(v)$. Suppose that $\pi_H(z_1) = \pi_H(z_2)$. Then we obtain: $Hgg_1G_e = Hgg_2G_e$, so $h = g(g_2g_eg_1^{-1})g^{-1}$, where $h \in H$, $g_e \in G_e \subseteq G_u$ (the last inclusion holds since we identify G_e with $\alpha_e(G_e)$). Therefore, the element $h \in H$ is conjugate to the element $g_2g_eg_1^{-1} \in G_u$, but H intersects trivially with the conjugates to all the vertex groups, in particular G_u. It follows that $h = 1$, so $gg_1G_e = gg_2G_e$. Therefore, $(H \cap K)gg_1G_e = (H \cap K)gg_2G_e$, so $z_1 = z_2$, and we get a contradiction. Thus, in this case $\pi_H(z_1) \neq \pi_H(z_2)$; similarly $\pi_K(z_1) \neq \pi_K(z_2)$.

Suppose now that $\omega(z_1) = \omega(z_2)$. Let $\omega(z_1) = \omega(z_2) = v$ correspond to $u \in V(Y)$, where $\omega(e) = u$, then, as mentioned above,

$$v = (H \cap K)gG_u, \quad z_1 = (H \cap K)gg_1t_e^{-1}G_e, \quad z_2 = (H \cap K)gg_2t_e^{-1}G_e, \quad g \in G,\ g_1, g_2 \in G_u.$$

Then, according to the definition of the projection π_H,

$$\pi_H(v) = HgG_u, \quad \pi_H(z_1) = Hgg_1t_e^{-1}G_e, \quad \pi_H(z_2) = Hgg_2t_e^{-1}G_e,$$

and $\omega(\pi_H(z_1)) = \omega(\pi_H(z_2)) = \pi_H(v)$. Suppose that $\pi_H(z_1) = \pi_H(z_2)$. Then we obtain: $Hgg_1t_e^{-1}G_e = Hgg_2t_e^{-1}G_e$, so $h = g(g_2t_e^{-1}g_et_eg_1^{-1})g^{-1}$, where $h \in H$, $g_e \in G_e$, $t_e^{-1}g_et_e \in G_u$

(the last inclusion holds since we identify G_e with $\alpha_e(G_e)$, so $t_e^{-1}G_et_e = \omega_e(G_e) \subseteq G_u$). Therefore, the element $h \in H$ is conjugate to the element $g_2(t_e^{-1}g_et_e)g_1^{-1} \in G_u$, but H intersects trivially with the conjugates to all the vertex groups, in particular G_u. It follows that $h = 1$, so $gg_1t_e^{-1}G_e = gg_2t_e^{-1}G_e$. Therefore, $(H \cap K)gg_1t_e^{-1}G_e = (H \cap K)gg_2t_e^{-1}G_e$, so $z_1 = z_2$, and we get a contradiction. Thus, in this case also $\pi_H(z_1) \neq \pi_H(z_2)$; similarly, $\pi_K(z_1) \neq \pi_K(z_2)$. ■

Lemma 2. *Suppose that the conditions of Theorem 1 hold. Let x, y be edges of the graphs T/H, T/K respectively. Then the number of edges of the graph $T/(H \cap K)$ which project under π_H into x and under π_K into y (simultaneously) is not bigger than m, where m is the maximum of the orders of the edge groups of (Γ, Y).*

□ We can assume that the edges x and y have the same orientation (both positive or both negative), since otherwise there are no edges of the graph $T/(H \cap K)$ which project under π_H into x and under π_K into y (simultaneously).

By considering if necessary x^{-1} and y^{-1} instead of x and y respectively, we can assume that both edges x and y are positively oriented.

Furthermore, we can assume that both edges x and y correspond to the same edge $e \in E(Y)^+$, since otherwise there are no edges of the graph $T/(H \cap K)$ which project under π_H into x and under π_K into y (simultaneously).

As mentioned above, we can suppose that

$$x = Hg_1G_e, \quad y = Kg_2G_e, \quad g_1, g_2 \in G.$$

Since $|G_e| \leqslant m$, it suffices to show that the number of edges of the graph $T/(H \cap K)$ which project under π_H into x and under π_K into y (simultaneously) is not bigger than $|G_e|$.

Suppose

$$z = (H \cap K)gG_e \in E(T/(H \cap K)) \ (g \in G) : \ \pi_H(z) = x, \ \pi_K(z) = y.$$

Then, according to the definition of the projections,

$$HgG_e = Hg_1G_e, \quad KgG_e = Kg_2G_e,$$

so

$$g = hg_1g_e = kg_2g_e',$$

where $g_e, g_e' \in G_e, h \in H, k \in K$. By considering $gg_e'^{-1}$ instead of g (and $g_eg_e'^{-1}$ instead of g_e), we can assume that $g_e' = 1$, so $g = hg_1g_e = kg_2$. It suffices to prove that for each (fixed) element $g_e \in G_e$ this condition determines a unique (if any) double coset $(H \cap K)gG_e$ (then for all $g_e \in G_e$ we obtain not more than $|G_e|$ edges $z \in E(T/(H \cap K))$ which project into x and y). Indeed, suppose that

$$g = hg_1g_e = kg_2, \quad g' = h'g_1g_e = k'g_2, \quad h, h' \in H, \ k, k' \in K.$$

Then we obtain:

$$g'g^{-1} = h'g_1g_eg_e^{-1}g_1^{-1}h^{-1} = h'h^{-1} \in H,$$
$$g'g^{-1} = k'g_2g_2^{-1}k^{-1} = k'k^{-1} \in K,$$

so $g'g^{-1} \in H \cap K$, therefore,

$$(H \cap K)g'G_e = (H \cap K)(g'g^{-1})gG_e = (H \cap K)gG_e.$$

Thus, Lemma 2 is proven. ■

Notice that if (supposing the conditions of Theorem 1 hold) the graph T/H is a tree, then, since $H \cong \pi_1(T/H)$, the subgroup H (and, therefore, $H \cap K$ as well) is trivial, and in this case the estimate (7), and thus Theorem 1, hold; the same is true if the graph T/K or

$T/(H\cap K)$ is a tree. Thus, we can assume below that the graphs T/H, T/K and $T/(H\cap K)$ are not trees.

Suppose a graph X is not a tree. We will call the *core* of the graph X a subgraph of X which consists of all vertices and edges of X which belong to any nontrivial closed cyclically reduced path in X. Notice that if a vertex $v \in V(X)$ belongs to the core of X then the core of X consists of all vertices and edges of X which belong to any reduced closed path in X beginning in v.

For a nontrivial subgroup $H \subseteq G$ (where G is the fundamental group of the graph of groups (Γ, Y)) denote by $\Psi(H)$ the core of the graph T/H. Notice that since T/H is connected $\Psi(H)$ is also connected. Notice also that the graph $\Psi(H)$ does not contain vertices of degree less than 2.

Lemma 3. *Suppose that the conditions of Theorem 1 hold (in particular, subgroup $H \subseteq G$ is finitely generated and intersects trivially with the conjugates to all the vertex groups of (Γ, Y)) and subgroup H is nontrivial. Then the graph $\Psi(H)$ is finite, $H \cong \pi_1(\Psi(H))$ and the following equalities hold:*

$$\overline{r}(H) = |E(\Psi(H))^+| - |V(\Psi(H))| = \frac{1}{2}\sum_{v\in V(\Psi(H))}(deg\, v - 2). \tag{11}$$

Similar statement holds for subgroups K and $H\cap K$ from Theorem 1.

$\square$ It was shown above that $H \cong \pi_1(T/H)$. Fix a vertex $v \in V(T/H)$ which lies in the subgraph $\Psi(H)$. Since any reduced closed path in T/H beginning in v lies in $\Psi(H)$, we obtain that $\pi_1(T/H, v) \cong \pi_1(\Psi(H), v)$, therefore, $H \cong \pi_1(\Psi(H))$.

Suppose reduced paths $p_1, ..., p_n$ are free generators of the group $\pi_1(\Psi(H), v)$; n is finite, since $H \cong \pi_1(\Psi(H))$ and H is finitely generated. Any reduced closed path in $\Psi(H)$ beginning in v is a product of some paths from $p_1, ..., p_n$ and their inverses. As mentioned above, any edge e of the graph $\Psi(H)$ belongs to some closed reduced path in $\Psi(H)$ beginning in v, so e belongs to at least one of the paths $p_1, ..., p_n$ and their inverses. Thus, the graph $\Psi(H)$ is finite.

Moreover, according to (3), we obtain:

$$r(H) = |E(\Psi(H))^+| - |V(\Psi(H))| + 1.$$

Therefore, the first equality in (11) holds.

Finally, the sum of the degrees of all vertices of any (oriented) graph is equal to the doubled number of its positively oriented edges, therefore, the second equality in (11) holds as well.

It is obvious that the same proof is true for the subgroups K and $H \cap K$ from Theorem 1. $\blacksquare$

Lemma 4. *Suppose that the conditions of Theorem 1 hold. Then the image of the graph $\Psi(H\cap K)$ under the projection π_H, π_K lies in the graph $\Psi(H)$, $\Psi(K)$ respectively. Thus, we can consider the restriction of the projections $\pi_H : \Psi(H\cap K) \to \Psi(H)$, $\pi_K : \Psi(H\cap K) \to \Psi(K)$.*

$\square$ The statement of this lemma follows immediately from Lemma 1. Indeed, suppose v is a vertex of the graph $\Psi(H\cap K)$. Then v belongs to some closed cyclically reduced path p in $T/(H\cap K)$. Since π_H is locally injective (Lemma 1) the closed path $\pi_H(p)$ in T/H is also cyclically reduced, and $\pi_H(v)$ belongs to this path, therefore, $\pi_H(v)$ belongs to the graph $\Psi(H)$. Similarly $\pi_K(v)$ belongs to the graph $\Psi(K)$. $\blacksquare$

Lemma 5. *Suppose that the conditions of Theorem 1 hold. Let a, b be vertices of the graphs $\Psi(H)$, $\Psi(K)$ respectively which correspond to the same vertex of the graph Y. Let $w_1, ..., w_s$*

be all vertices of the graph $\Psi(H \cap K)$ which project under π_H into a and under π_K into b. Then the following inequalities hold:

$$deg\, w_i \leqslant deg\, a, \quad deg\, w_i \leqslant deg\, b, \quad i = 1, ..., s \tag{12}$$

$$\sum_{i=1}^{s} deg\, w_i \leqslant m \cdot deg\, a \cdot deg\, b, \tag{13}$$

where m is the maximum of the orders of the edge groups of (Γ, Y).

□ Due to Lemma 4 each edge of the graph $\Psi(H \cap K)$ beginning in one of the vertices w_i $(i = 1, ..., s)$ projects under π_H into an edge of the graph $\Psi(H)$ beginning in a, and projects under π_K into an edge of the graph $\Psi(K)$ beginning in b.

Inequality (12) now follows immediately from Lemma 1.

Applying Lemma 2 for every edge x of the graph $\Psi(H)$ beginning in a and for every edge y of the graph $\Psi(K)$ beginning in b, we obtain inequality (13) as well. ■

We will now complete the proof of Theorem 1. The following part of the proof follows the idea of S.V.Ivanov [4].

Applying the equalities (11) from Lemma 3, we can reformulate the estimate (7) of theorem 1 in terms of the degrees of vertices of the graphs Ψ:

$$\sum_{w \in V(\Psi(H \cap K))} (deg\, w - 2) \leqslant 3m \cdot \sum_{a \in V(\Psi(H))} (deg\, a - 2) \cdot \sum_{b \in V(\Psi(K))} (deg\, b - 2) \tag{14}$$

Thus, to prove Theorem 1 it suffices to prove the inequality (14). Notice that to prove the inequality (14) it suffices to prove the following inequality:

$$\sum_{i=1}^{s_{a,b}} (deg\, w_i^{a,b} - 2) \leqslant 3m \cdot (deg\, a - 2) \cdot (deg\, b - 2), \tag{15}$$

for all vertices $a \in V(\Psi(H))$, $b \in V(\Psi(K))$ such that a corresponds to the same vertex of the graph Y as b. Here $w_1^{a,b}, ..., w_{s_{a,b}}^{a,b}$ are all vertices of the graph $\Psi(H \cap K)$ which project under π_H into a and under π_K into b.

Indeed, suppose the inequality (15) holds. Then, according to Lemma 4, we obtain:

$$\sum_{w \in V(\Psi(H \cap K))} (deg\, w - 2) = \sum_{(a,b)} \sum_{i=1}^{s_{a,b}} (deg\, w_i^{a,b} - 2) \leqslant \sum_{(a,b)} 3m \cdot (deg\, a - 2) \cdot (deg\, b - 2) \leqslant$$

$$\leqslant 3m \cdot \sum_{a \in V(\Psi(H))} (deg\, a - 2) \cdot \sum_{b \in V(\Psi(K))} (deg\, b - 2),$$

where the sum $\sum_{(a,b)}$ extends over all vertices $a \in V(\Psi(H))$, $b \in V(\Psi(K)$ such that a corresponds to the same vertex of the graph Y as b. (Here the first equality holds since, if $w \in V(\Psi(H \cap K))$ corresponds to the vertex $v \in V(Y)$, then the projections $\pi_H(w) \in V(\Psi(H))$ and $\pi_K(w) \in V(\Psi(K))$ correspond to the same vertex $v \in V(Y)$.) Thus, if (15) holds, then (14) holds as well.

It suffices to prove the inequality (15). We can assume without loss of generality that

$$deg\, a \leqslant deg\, b. \tag{16}$$

We see that the conditions of Lemma 5 hold; applying this lemma, we get the following inequalities:

$$deg\, w_i^{a,b} \leqslant deg\, a, \quad i = 1, ..., s_{a,b}, \tag{17}$$

$$\sum_{i=1}^{s_{a,b}} deg\, w_i^{a,b} \leqslant m \cdot deg\, a \cdot deg\, b. \tag{18}$$

Consider two cases. If $s_{a,b} \leqslant m \cdot deg\, b$, then, applying inequality (17), we obtain:

$$\sum_{i=1}^{s_{a,b}} (deg\, w_i^{a,b} - 2) \leqslant s_{a,b}(deg\, a - 2) \leqslant m \cdot deg\, b \cdot (deg\, a - 2).$$

If $s_{a,b} \geqslant m \cdot deg\, b$, then, applying inequality (18), we obtain:

$$\sum_{i=1}^{s_{a,b}} (deg\, w_i^{a,b} - 2) = \sum_{i=1}^{s_{a,b}} deg\, w_i^{a,b} - 2s_{a,b} \leqslant m \cdot deg\, a \cdot deg\, b - 2m \cdot deg\, b = m \cdot deg\, b \cdot (deg\, a - 2).$$

Thus, in any case the following inequality holds:

$$\sum_{i=1}^{s_{a,b}} (deg\, w_i^{a,b} - 2) \leqslant m \cdot deg\, b \cdot (deg\, a - 2). \tag{19}$$

Moreover, the graph $\Psi(H)$ has no vertices of degree less than 2. Therefore, according to (16), $deg\, b \geqslant deg\, a \geqslant 2$. If $deg\, b = 2$, then $deg\, a = 2$, so (19) implies (15). Otherwise, if $deg\, b \geqslant 3$, then $deg\, b \leqslant 3(deg\, b - 2)$, therefore,

$$m \cdot deg\, b \cdot (deg\, a - 2) \leqslant 3m \cdot (deg\, a - 2) \cdot (deg\, b - 2),$$

so (19) again implies (15). Therefore, in any case the inequality (15) holds.

Thus, Theorem 1 is proven.

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